An Interactive Introduction to Complex Numbers

3. Exponentiations

Example 3.1:

Open the Applet Exponentiations and set $z = \sqrt{2} \cdot e^{22.5° \cdot i}$ in exponential form. Then click on z^n and set $n = 2$. Look at the result in exponential form. Note that $\left|z^2\right| = |z|^2 = r^2$ and $\arg\left(z^2\right) = 2 \cdot \varphi$.

Indeed, we calculate $z^2 = \left(\sqrt{2} \cdot e^{22.5° \cdot i}\right)^2 = \left(\sqrt{2}\right)^2 \cdot \left(e^{22.5° \cdot i}\right)^2 = 2 \cdot e^{22.5° \cdot i \cdot 2} = 2 \cdot e^{45° \cdot i}$.

Shift n to $n = 4$, which leads to $z^4 = \left(\sqrt{2}\right)^4 \cdot \left(e^{22.5° \cdot i}\right)^4 = 4 \cdot e^{22.5° \cdot i \cdot 4} = 4 \cdot e^{90° \cdot i}$.

Rule 3.1

The n-th exponentiation of $z = r \cdot e^{\varphi \cdot i}$ in exponential form is $z^n = \left(r \cdot e^{\varphi \cdot i}\right)^n = r^n \cdot \left(e^{\varphi \cdot i}\right)^n = r^n \cdot e^{n \cdot \varphi \cdot i}$ $\quad n \in \mathbb{R}$ with $\left|z^n\right| = |z|^n = r^n$ and $\arg\left(z^n\right) = n \cdot \varphi$.

There exists no general rule for the exponentiation of $z = a + b \cdot i$ in cartesian form. If $n \in \mathbb{N}$, we could use the general binomial theorem. However, in most cases it is easier to transform $z = a + b \cdot i$ into exponential form and apply rule 3.1.

In case of $n = 2$ the first binomial equation tells us that $z^2 = (a + b \cdot i)^2 = a^2 + 2 \cdot a \cdot b \cdot i + (b \cdot i)^2 = a^2 - b^2 + 2 \cdot a \cdot b \cdot i$.

Recall, we defined the exponent as $n \in \mathbb{R}$, i. e. negative values of n are feasible as well.

Example 3.2:

Open the Applet Exponentiations and set $z = \sqrt{2} \cdot e^{22.5° \cdot i}$ in exponential form. Then click on z^n and set $n = -2$. Of course, we get $z^{-2} = \left(\sqrt{2} \cdot e^{22.5° \cdot i}\right)^{-2} = \left(\sqrt{2}\right)^{-2} \cdot \left(e^{22.5° \cdot i}\right)^{-2}$ $= 0.5 \cdot e^{22.5° \cdot i \cdot (-2)} = 0.5 \cdot e^{-45° \cdot i} = 0.5 \cdot e^{315° \cdot i}$.

The result can also be derived by calculating $z^{-2}=\frac{1}{z^2}$ just as with real numbers.

Regarding example 3.1 we calculate $z^{-2}=\frac{1}{z^2}=\frac{1}{2\cdot e^{45°\cdot i}}=\frac{1\cdot e^{0°\cdot i}}{2\cdot e^{45°\cdot i}}=0.5\cdot e^{-45°\cdot i}=0.5\cdot e^{315°\cdot i}$.

Rule 3.2

$z^{-n}=\frac{1}{z^n} \quad n\in\mathbb{R}, \mathrm{Re}(z)\neq 0 \vee \mathrm{Im}(z)\neq 0, |z|\neq 0$

Roots

The natural question to ask now is, whether $z=r\cdot e^{\varphi\cdot i}$ has also roots $w=\sqrt[k]{z}$ i.e. are there any complex numbers w solving for $w^k=z \quad k\in\mathbb{N}$?

Example 3.3 a

From example 3.1 we can conclude that $w=\sqrt{2}\cdot e^{22.5°\cdot i}=1.4142\cdot e^{22.5°\cdot i}$ is a square root of $z=2\cdot e^{45°\cdot i}$ or, alternatively, a solution of $w^2=2\cdot e^{45°\cdot i}$. But is it the only one? To check this out, open the Applet Exponentiations and set $z=2\cdot e^{45°\cdot i}$ in exponential form. Then click on $w^k=z$ and set $k=2$. We see that $w^2=2\cdot e^{45°\cdot i}$ has two solutions: $w_1=1.4142\cdot e^{22.5°\cdot i}$ and $w_2=1.4142\cdot e^{202.5°\cdot i}$.

What is the intuition behind the second solution w_2? Recall that $z=2\cdot e^{45°\cdot i}$ corresponds to $z=2\cdot e^{405°\cdot i}$. Then $w_2=\left(2\cdot e^{405°\cdot i}\right)^{0.5}=\sqrt{2}\cdot e^{405°\cdot i\cdot 0.5}=1.4142\cdot e^{202.5°\cdot i}$.

$z=2\cdot e^{45°\cdot i}$ also corresponds to $z=2\cdot e^{765°\cdot i}$, $z=2\cdot e^{1'125°\cdot i}$ and so on. But why do further solutions not exist? We see that $w=\left(2\cdot e^{765°\cdot i}\right)^{0.5}=1.4142\cdot e^{382.5°\cdot i}$, which corresponds to $w_1=1.4142\cdot e^{22.5°\cdot i}$, whereas $w=\left(2\cdot e^{1'125°\cdot i}\right)^{0.5}=1.4142\cdot e^{562.5°\cdot i}$ corresponds to $w_2=1.4142\cdot e^{202.5°\cdot i}$.

Example 3.3 b:

Open the Applet Exponentiations and set $z=2\cdot e^{45°\cdot i}$ in exponential form. Then click on $w^k=z$ and set $k=3$ and $k=4$.

For $k=3$ we get $w_1=1.2599\cdot e^{15°\cdot i}$, $w_2=1.2599\cdot e^{135°\cdot i}$ and $w_3=1.2599\cdot e^{255°\cdot i}$.

For $k=4$ we get $w_1=1.1892\cdot e^{11.25°\cdot i}$, $w_2=1.1892\cdot e^{101.25°\cdot i}$, $w_3=1.1892\cdot e^{191.25°\cdot i}$ and $w_4=1.1892\cdot e^{281.25°\cdot i}$.

Rule 3.3 a

$w^k=z=r\cdot e^{\varphi\cdot i}$ has $k\in\mathbb{N}$ solutions. The angle between succeeding solutions is $\frac{360°}{k}$ (degree) or $\frac{2\cdot\pi}{k}$ (radian).

Rule 3.3 b

The first solution of $w^k=z=r\cdot e^{\varphi\cdot i}$ is $w_1=\sqrt[k]{r}\cdot e^{\frac{\varphi}{k}\cdot i}$.

Rule 3.3 c

In degree the j-th solution of $w^k=z=r\cdot e^{\varphi\cdot i}$ is $w_j=\sqrt[k]{r}\cdot e^{\frac{\varphi+(j-1)\cdot 360°}{k}\cdot i}$ with $|w_j|=\sqrt[k]{r}$ and $\varphi_{wj}=\arg w_j=\frac{\varphi+(j-1)\cdot 360°}{k}$.

In radian the j-th solution of $w^k=z=r\cdot e^{\varphi\cdot i}$ is $w_j=\sqrt[k]{r}\cdot e^{\frac{\varphi+(j-1)\cdot 2\cdot\pi}{k}\cdot i}$ with $|w_j|=\sqrt[k]{r}$ and $\varphi_{wj}=\arg w_j=\frac{\varphi+(j-1)\cdot 2\cdot\pi}{k}$.

There is no general rule for roots of complex numbers in cartesian form.

Exercise 3.1:

We have $z=1.5\cdot e^{117°\cdot i}$ and $z=5-2\cdot i$. Determine z^2, z^{-3} and $z^{0.6}$ for both numbers. Use the Applet Exponentiations to check your answers.

Exercise 3.2:

Turn $z=5\cdot i$ into exponential form. Then determine all solutions of $w^5=z$ and rewrite them in cartesian form. Use the Applet Exponentiations to check your answers.